8. BOUNDARY LAYER THEORY

The complete approximate solution to the Navier-Stokes equations may be built up from two part solutions valid in different regions of the flow field. One of these is the solution of the inviscid flow problem, the so-called *outer solution*, and the other is the *inner solution* close to the wall. The inner solution describes the *boundary layer flow* and must such that the flow velocity from its value zero at the wall passes asymptotically into the velocity predicted by the outer (inviscid) solution directly at the wall. We will show that the thickness of the boundary layer, i.e. the layer where the friction effects cannot be ignored, is proportional to $\mathrm{Re}^{-1/2}$. The inviscid solution then represents an approximate solution of the Navier-Stokes equations for large Reynolds number, with an error of order $O(\mathrm{Re}^{-1/2})$.

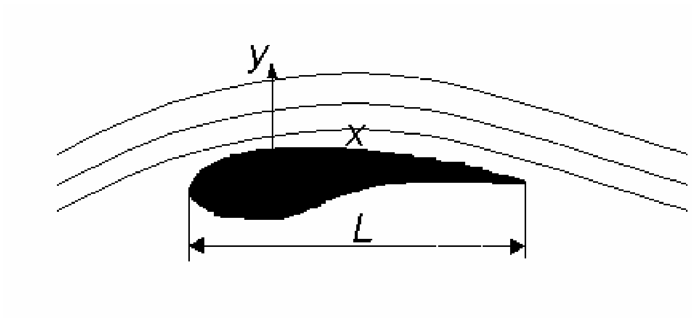

Figure 8.1. Boundary layer coordinates

Consider an incompressible and plane two-dimensional flow. We introduce the so-called boundary layer coordinate system, in which x is measured along the surface of the body and y perpendicular to it. If the boundary layer thickness is small compared to the radius of curvature R of the wall contour $(\delta / R \ll 1)$, the Navier-Stokes equations hold in the same form as in cartesian coordinates. In the calculation of the inner solution, i.e. of the boundary layer flow, the curvature of the wall then plays no role. The boundary layer develops as if the wall were flat. The wall curvature only manifests itself indirectly through the pressure distribution given by the outer solution.

Since the boundary layer is very thin for large Reynolds numbers, the following inequalities hold:

$$\frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial y^2}.$$

A consequence of the last condition is that the x component of the Navier-Stokes equations reduces to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}.$$

In order to determine the order of magnitude of the term $u\dfrac{\partial u}{\partial x}$ in comparison to $v\dfrac{\partial u}{\partial y}$, we begin with the continuity equation for plane two-dimensional and incompressible flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

and together with (8.1) conclude that $\partial v/\partial y \ll \partial u/\partial y$, so that $v \ll u$ holds. Therefore the second and third terms on the left hand side in (8.2) are of the same order of magnitude.

While the viscous forces are completely ignored in the outer flow, they do play a role in the boundary layer. The order of magnitude of the boundary layer thickness can be determined by considering the thickness of the layer where the viscous forces are of the same order of magnitude as the inertial forces, e.g. where

$$\frac{u}{\nu}\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial^2 u}{\partial y^2}} \sim 1 .$$

In the x direction, let L be the typical length scale (Figure 8.1), and if U_∞ is the incident flow velocity, we have the order of magnitude equation

$$u\frac{\partial u}{\partial x} \sim \frac{U_\infty^2}{L} .$$

The typical length scale in the y direction is the average boundary layer thickness δ_0, so that

$$\nu\frac{\partial^2 u}{\partial y^2} \sim \nu\frac{U_\infty}{\delta_0^2} .$$

We then have the estimate

$$\frac{\dfrac{U_\infty^2}{L}}{\nu\dfrac{U_\infty}{\delta_0^2}} \sim 1 ,$$

from which we obtain

$$\frac{\delta_0}{L} \sim \mathrm{Re}^{-1/2} .$$

With this result, the individual terms in the equations of motion are reviewed in order to systematically simplify the equations themselves. It follows from the continuity equation that

$$v \sim \frac{\delta_0}{L}U_\infty \quad \text{and therefore} \quad v \sim U_\infty \,\mathrm{Re}^{-1/2} .$$

We now introduce the dimensionless quantities, chosen so that they are all of the same order of magnitude

$$u^* = \frac{u}{U_\infty} ,\quad v^* = \frac{v}{U_\infty}\frac{L}{\delta_0} = \frac{v}{U_\infty}\mathrm{Re}^{1/2} ,\quad p^* = \frac{p}{\rho U_\infty^2}$$

and

$$x^* = \frac{x}{L} ,\quad y^* = \frac{y}{\delta_0} = \frac{y}{L}\mathrm{Re}^{1/2} ,\quad t^* = t\frac{U_\infty}{L} .$$

Using these variables the Navier-Stokes equations take on the form

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{\mathrm{Re}}\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}$$

and

$$\frac{1}{\mathrm{Re}}\left(\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)=-\frac{\partial p^*}{\partial x^*}+\frac{1}{\mathrm{Re}^2}\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{1}{\mathrm{Re}}\frac{\partial^2 u^*}{\partial y^{*2}}$$

in which all differential expressions have the same order of magnitude, and the order of magnitude of the whole term is controlled by the prefactor.

Since we are looking for an approximate solution for large Reynolds numbers, we take the limit $\mathrm{Re}\to\infty$ and obtain the *boundary layer equations* in dimensionless form:

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{\partial^2 u^*}{\partial y^{*2}},$$

and

$$0=-\frac{\partial p^*}{\partial y^*}.$$

In addition we have the continuity equation which remained unaffected by taking the limit

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0.$$

The dynamic boundary condition at the wall reads

$$\text{for}\quad y^*\to\infty,\qquad u^*\to\frac{U}{U_\infty}.$$

In the dimensionless boundary layer equations and in the boundary conditions the viscosity does not appear, and therefore the solution is valid for all Reynolds numbers, as long as they are large enough.

We shall now rewrite the boundary layer equations in dimensional form and shall restrict ourselves to steady flow. These were first stated in this form in 1904 by Prandtl:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2},$$

$$0=\frac{\partial p}{\partial y},$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

From the second equation of this system of partial differential equations of the parabolic type we see that $p=p(x)$. In the remaining equations u and v are the independent variables, while p is no longer to be counted as an unknown because the pressure in the boundary layer $p(x)$ has the same value as outside it, where it is known from the outer solution. Therefore the pressure gradient is a known function, and using Euler equation, can be replaced by

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = U\frac{\partial U}{\partial x}.$$

We also note that for $y \to \infty$ only one condition is placed, on the component u. The boundary conditions reads as

$$\text{for} \quad x = x_0, \quad u = u_0(y)$$

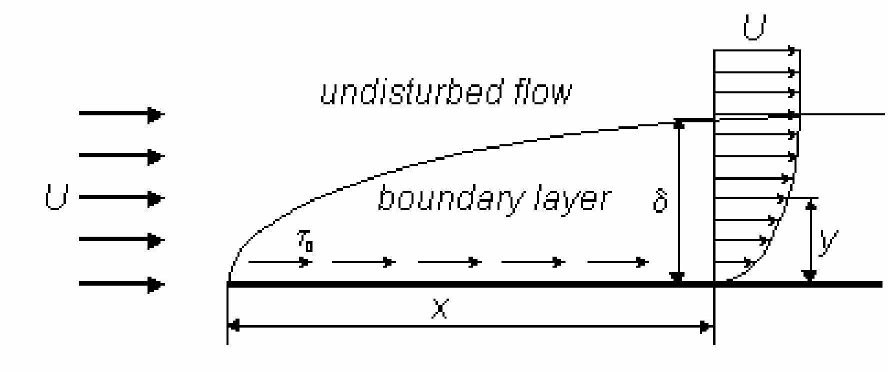

Figure 8.2. Growth of boundary layer along a smooth plate

8.1. ***Friction drag of boundary layer***

Figure 8.2 shows the growth of the boundary layer along one side of a smooth plate in steady flow of an incompressible fluid. Let us consider the control volume shown in Figure 8.3 which extends a distance δ from the plate, where δ is the thickness of the boundary layer at a distance x along the plate. Here we use δ to indicate the thickness of the boundary layer, usually defined as the distance from the boundary to the point where the velocity u = 0.99U. In this analysis, however, we will assume that $u = U$ at the edge of the boundary layer. Along control surface AB the undisturbed velocity U exists. The pressure forces around the periphery of the control volume will cancel one another out since the undisturbed flow field pressure must exist along AB and DA, and the distance BC (=δ) is so small that it will have a negligible effect on pressure variations.

Applying the momentum-impulse theorem we get

$$\begin{aligned} -F_x = -\text{drag} &= \text{momentum leaving through } BC \\ &+ \text{momentum leaving through } AB \\ &- \text{momentum entering through } DA \end{aligned}$$

Since $Q_{BC} < Q_{DA}$, there is flow out of the control volume across control surface AB and $Q_{AB} = Q_{DA} - Q_{BC}$.

If the width of the plate is B, neglecting edge effects, the flows and momentums across the control surface can be expressed as follows:

Control surface	Flow	Momentum
DA	$UB\delta$	$\rho(UB\delta)U$
BC	$B\int_0^\delta u dy$	$\rho B\int_0^\delta u^2 dy$
AB	$UB\delta - B\int_0^\delta u dy$	$\rho\left(UB\delta - B\int_0^\delta u dy\right)U$

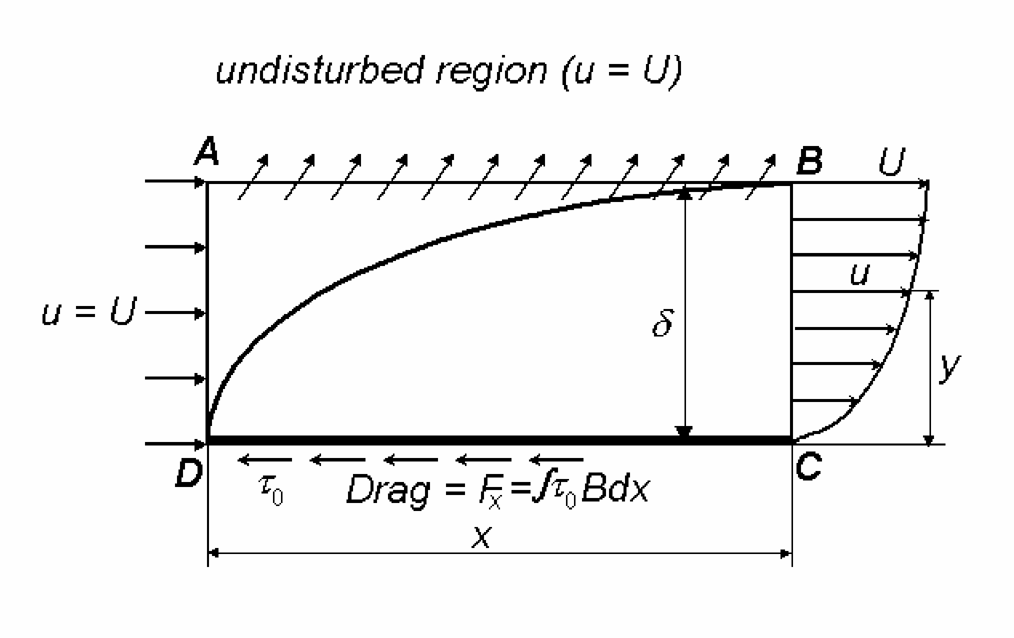

Figure 8.3. Control volume for flow over one side of a flat plate

Substituting these momentum values in momentum-impulse theorem equation gives

$$F_x = \rho B\int_0^\delta u(u-U)dy$$

where F_x is the total friction rag of the plate on the fluid *from the leading edge up to x* directed to the left, as shown in Figure 8.3. Equal and opposite to this is the drag of the fluid on the plate.

It will now be assumed that the velocity profiles at various distances along the plate are similar to each other:

$$\frac{u}{U} = f\left(\frac{y}{\delta}\right) = f(\eta).$$

There is experimental evidence that this assumption is valid if there is no pressure gradient along the surface and if the boundary layer does not change from laminar to turbulent within the region considered. Then, substituting for u in the last equation and changing the variable y to the dimensionless η, $dy = \delta d\eta$, and the limits become = to 1, giving

$$F_x = \rho BU^2\delta\int_0^1 f(\eta)[1-f(\eta)]\,d\eta,$$

which, for convenience, may be written

$$F_x = \rho BU^2\delta\alpha$$

where α is a function of the boundary layer velocity distribution only and is given by the indicated integral.

We next investigate the local wall shear stress τ_0 at distance x from the leading edge. From the definition of surface resistance, $dF_x = \tau_0 B dx$, or

$$\tau_0 = \frac{1}{B}\frac{dF_x}{dx} = \frac{1}{B}\frac{d}{dx}\left(\rho B U^2 \delta \alpha\right)$$

and as all terms in the expression for F_x are constant except δ,

$$\tau_0 = \rho U^2 \alpha \frac{d\delta}{dx}.$$

This expression for the shear stress is valid for either laminar or turbulent flow in the boundary layer, but in this form it is not useful until the quantities α and $d\delta/dx$ are evaluated.

8.2. *Laminar boundary layer for incompressible flow along a smooth flat plate*

As in the case of laminar flow in pipes, we may examine the shear stress at the plate by aid of velocity gradient and the definition of viscosity (here we introduce μ as the dynamic viscosity of the fluid),

$$\tau_0 = \mu\left(\frac{du}{dy}\right)_{y=0} = \frac{\mu}{\delta}\left(\frac{du}{d\eta}\right)_{\eta=0} = \frac{\mu U}{\delta}\left(\frac{df(\eta)}{d\eta}\right)_{\eta=0},$$

which may be abbreviated to

$$\tau_0 = \frac{\mu U \beta}{\delta},$$

where β, like α, is a dimensionless function of the velocity distribution curve and is given by the expression in brackets.

Equating the two last expressions for τ_0 results in a simple differential equation

$$\delta\, d\delta = \frac{\mu\beta}{\rho U \alpha} dx$$

with the solution

$$\frac{\delta^2}{2} = \frac{\mu\beta}{\rho U \alpha} x + C$$

where $C = 0$, since $\delta = 0$ at $x = 0$. Therefore,

$$\delta = \sqrt{\frac{2\mu\beta x}{\rho U \alpha}} = \sqrt{\frac{2\beta}{\alpha}}\frac{x}{\sqrt{\mathrm{Re}}},$$

where $\mathrm{Re} = xU\rho/\mu$ may be called the local Reynolds number. It should be noted that Re increases linearly in the downstream direction. Examination of the first expresion of the above equation shows that the thickness of the laminar boundary layer increases with distance from the leading edge; thus the shear stress decreases as the layer grows along the plate.

To evaluate the last equation, we must know or assume the velocity profile in the laminar boundary layer. The velocity distribution may be closely represented by a parabola, as shown in Figure 8.4. In dimensionless terms we have

$$\frac{u}{U}=f(\eta)=2\eta-\eta^2 .$$

The other velocity profile in Figure 8.4 was derived by Blasius from the fundamental equations of viscous flow. This curved is based on the thickness δ being defined for which $u = 0.99U$.

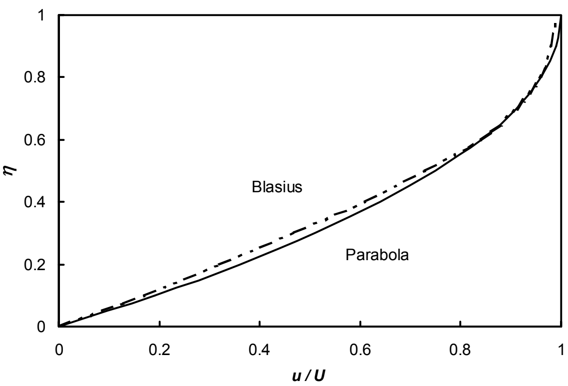

Figure 8.4. Velocity distribution in laminar boundary layer on a flat plate

The parabolic distribution gives numerical values for α and β of 0.133 and 2, rspectively. The Blasius curve yields $\alpha = 0.135$ and $\beta = 1.63$, the principal difference lying in the milder slope of the velocity gradient at the wall. With the Blasius values substituted in the expression of δ we obtain

$$\frac{\delta}{x}=\sqrt{\frac{2\times 1.63}{0.135}}\frac{1}{\sqrt{\mathrm{Re}}}=\frac{4.91}{\sqrt{\mathrm{Re}}} .$$

Instead of the geometric boundary layer thickness δ, the uniquely defined *displacement thickness* δ_1 is often preferred:

$$\delta_1=\int_0^\infty\left(1-\frac{u}{U}\right)dy ,$$

which is a measure of the displacing action of the boundary layer. We obtain

$$\frac{\delta_1}{x}=\frac{1.7208}{\sqrt{\mathrm{Re}}} .$$

A measure for the loss of momentum in the boundary layer is the *momentum thickness* δ_2:

$$\delta_2=\int_0^\infty\left(1-\frac{u}{U}\right)\frac{u}{U}dy ,$$

for which we obtain the value

$$\frac{\delta_1}{x}=\frac{0.664}{\sqrt{\mathrm{Re}}} .$$

If the value of δ from $\delta/x = 4.91/\sqrt{\mathrm{Re}}$ is substituted in the expression $\tau_0 = \mu U\beta/\delta$ with $\beta = 1.63$, there results for the shear stress

$$\tau_0 = 0.322\frac{\mu U}{x}\sqrt{\mathrm{Re}}\,.$$

If the boundary layer remains laminar over the length of L of the plate, the total friction drag on one side of the plate is given by integrating this expression:

$$F_f = B\int_0^L \tau_0 dx = 0.322B\sqrt{\rho\mu U^3}\int_0^L x^{-1/2}dx = 0.664B\sqrt{\rho\mu L U^3}\,,$$

and the coefficient of friction c_f may be obtained as

$$c_f = \frac{F_f}{\dfrac{\rho U^2 L}{2}} = \frac{1.328}{\sqrt{\mathrm{Re}}}\,,$$

a result which is called *Blasius friction law*.

The laminar boundary layer will remain laminar if undisturbed, up to a value of Re of about 500,000. In this region the layer becomes turbulent, increasing noticeably in thickness and displaying a marked change in velocity distribution (Figure 8.5).

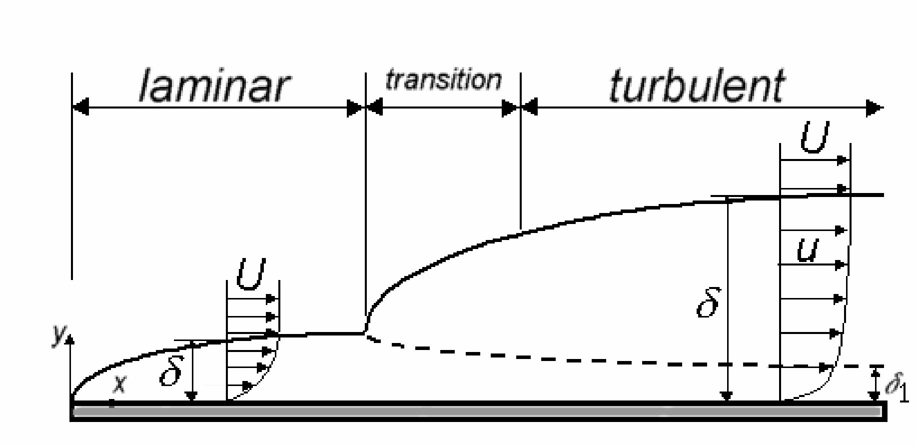

Figure 8.5. Laminar and turbulent boundary layers along a smooth flat plate

The thickness of the turbulent boundary layer (which also contains a viscous sublayer of thickness δ_1) is

$$\frac{\delta}{x} = \frac{0.377}{\mathrm{Re}^{1/5}}\,,$$

while the coefficient of friction have the expressions

$$c_f = \frac{0.0735}{\mathrm{Re}^{1/5}} \qquad \text{for} \qquad \mathrm{Re} < 10^7,$$

and

$$c_f = \frac{0.455}{(\log \mathrm{Re})^{2.58}} \qquad \text{for} \qquad \mathrm{Re} > 10^7.$$

8.3. *Boundary-layer separation*

The motion of a thin stratum of fluid lying wholly inside the boundary layer is determined by three forces:

1. The forward pull of the outer free-moving fluid transmitted through the laminar boundary layer by viscous shear and through the turbulent boundary layer by momentum transfer.
2. The viscous retarding effect of the solid boundary which must, by definition, hold the fluid stratum immediately adjacent to it at rest.
3. The pressure gradient along the boundary. The stratum is accelerated by a pressure gradient whose pressure decreases in the direction of flow and is retarded by an adverse gradient.

The treatment of fluid resistance in the foregoing sections has been restricted to the drag of the boundary layer along a smooth flat plate located in an unconfined fluid, that is to say, in the absence of a pressure gradient.

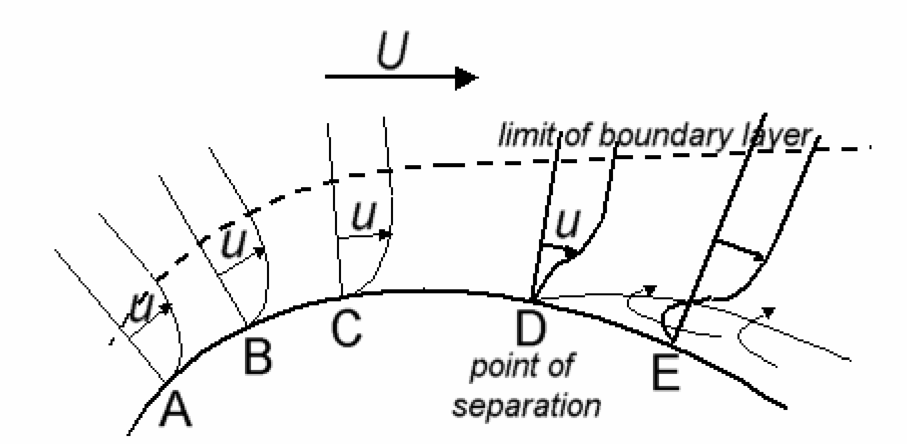

Figure 8.6. Growth and separation of boundary layer owing to increasing pressure gradient. Note that U has its maximum value at B and then gets smaller.

In the presence of a favorable pressure gradient the boundary layer is held in place. This is what occurs in the accelerated flow around the forebody, or upstream portion, of a cylinder, sphere, or other object. If a particle enters the boundary layer near the forward stagnation point with a low velocity and high pressure, its velocity will increase as it flows into the lower pressure region along the side of the body. But there will be some retardation from wall friction so that its total energy will be reduced by a corresponding conversion into thermal energy.

Consider the flow around a body depicted in Figure 8.6. Let *A* represent a point in the region of accelerated flow, with a normal velocity distribution in the boundary layer, while *B* is the point where the velocity outside the boundary layer reaches a maximum. Then *C*, *D*, and *E* are points downstream where the velocity outside the boundary layer decreases, resulting in an increase in pressure. Thus the velocity of the layer close to the wall is reduced at C and finally brought up to a stop at *D*. Now the increasing pressure calls for further retardation. However, this is impossible, and so the boundary layer actually separates from the wall. At *E* there is a backflow next to the wall, driven in the direction of decreasing pressure and feeding fluid into the boundary layer which has left the wall at *D*.

Downstream from the point of separation the flow is characterized by irregular turbulent eddies, formed as the separated boundary layer becomes rolled up in the reversed flow. This condition generally extends for some distance downstream until the eddies are worn away by viscous attrition.

Because the eddies cannot convert their kinetic energy of rotation into an increase pressure, the pressure within the wake remains close to that at the separation point. Since this is always less than the pressure at the forward stagnation point, there results a net pressure difference tending o move the body with the flow, and this force is the pressure drag.